# Condensation in zero-range processes on inhomogeneous networks

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We investigate the role of inhomogeneities in zero-range processes in condensation dynamics. We consider the dynamics of balls hopping between nodes of a network with one node of degree $k_1$ much higher than a typical degree $k$, and find that the condensation is triggered by the inhomogeneity and that it depends on the ratio $k_1/k$. Although, on the average, the condensate takes an extensive number of balls, its occupation can oscillate in a wide range. We show that in systems with strong inhomogeneity, the typical melting time of the condensate grows exponentially with the number of balls.

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## I. INTRODUCTION

Zero-range processes [1–6] have attracted attention of many researchers since they provide an exactly solvable example of far-from-equilibrium dynamics and of condensate formation. Many questions concerning the dynamics of the model can be addressed and solved analytically. It is known that a zero-range process has a steady state and that static properties of this state are described by the partition function of the balls-in-boxes (B-in-B) model [7,8], a generalization of the Backgammon model [9]. The B-in-B model has two phases: a fluid and a condensed one, separated by a critical point at which the system undergoes a phase transition and the condensate is formed. Unlike the Bose-Einstein condensation, the B-in-B condensation takes place in real space rather than in momentum space. Therefore it mimics such processes like mass transport [1], condensation of links in complex networks [2,10] or phase separation [11,12].

The zero-range process (ZRP) [13] discussed in this paper describes a gas of identical, indistinguishable particles hopping between the neighboring nodes of a network. The state of such a system is characterized by the topology of the network, which is fixed during the process, and by the particle distribution which is given by the occupation numbers of particles $\{m_i\}$ on all nodes $i=1,\ldots,N$ of the network. The total number of particles $M=m_1+m_2+\cdots+m_N$ is conserved during the process. The zero-range dynamics is characterized by the outflow rates $u(m)$ of particles from network nodes, which depend only on the occupation number $m$ of the node from which the particle hops. We shall assume that these ultralocal hopping rates are identical for each node. The stream of particles outgoing from a node is equally distributed among all its neighbors. So if we denote by $k_i$ the number of neighbors of the node $i$, called also its degree, then the hopping rate from the $i$th node to each of its neighbors is equal to $u(m_i)/k_i$. The outflow rate $u(m)$ is a semipositive function which is equal to zero for $m=0$. For a given graph the function $u(m)$ entirely defines the dynamics of the system.

We consider here the ZRP on a network being a connected simple graph. In this case, the ZRP has a unique steady state, in which the probability $P(m_1,\ldots,m_N)$ of finding the distribution of balls $\{m_1,\ldots,m_N\}$ factorizes into a product of some weight functions $p_i(m_i)$ for individual nodes, except that there is a global constraint reflecting the conservation of particles.

On a $k$-regular network, that is when all node degrees are equal to $k$, all weight functions $p_i(m)$ are identical, $p_i(m)\equiv p(m)$, and thus the probability $P$ is invariant under permutations of the occupation numbers. When the density $\rho=M/N$ of balls per node exceeds a certain critical value $\rho_c$ depending on the functional form of $p(m)$, a single node attracts an extensive number of balls called the condensate. The relative occupation of that node does not disappear in the thermodynamic limit $N,M\to\infty$, with fixed $\rho$. The larger the density $\rho$, the larger is the number of balls in the condensate. In other words, the system undergoes a phase transition at $\rho=\rho_c$ between the fluid (low density) and the condensed (high density) phase. The permutation symmetry of $\{m_i\}$ is respected in the fluid phase while it is broken in the condensed phase where one node becomes evidently distinct from the $N-1$ remaining ones. The symmetry of the partition function reduces to the subgroup of permutations of $N-1$ occupation numbers. This mechanism has been extensively studied in the B-in-B model [7,8,14]. The value of the critical density depends on the weight function $p(m)$ which can be translated to the asymptotic properties of $u(m)$ for $m\to\infty$. If $u(m)$ tends to infinity then also $\rho_c=\infty$ and the condensation does not occur regardless of the density of balls. The system is in the fluid phase for any finite density $\rho$. Intuitively, this means that there exists an effective repulsive force preventing a node from being occupied by many balls and they distribute uniformly on the whole graph. On the contrary, if $u(m)\to 0$, the critical density is $\rho_c=0$ and therefore the system is in the condensed phase for any $\rho>0$. The larger the number of particles on a node, the smaller are the

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chances for balls to escape from it since $u(m)$ becomes very small for large $m$. This can be seen as the existence of an effective attraction between particles.

The most interesting case is when $u(m)$ goes to some positive constant $u_\infty$ with $m \to \infty$. One can show that the probability distribution $P(m_1, \ldots, m_N)$ does not depend on $u_\infty$ but on how fast $u(m)$ approaches the constant value. Therefore, without loss of generality we can choose $u_\infty = 1$ and concentrate on the asymptotic behavior having the form $u(m) = 1 + b/m$, with $b$ being some positive number. If $0 \leq b \leq 2$, then the critical density $\rho_c$ is infinite. The effective attraction between balls is too weak to form the condensate. However, if $b > 2$, then $\rho_c$ has a finite value. In this case the attraction is strong enough to trigger the condensation above the critical density $\rho_c$.

So far we have discussed the criteria of the condensation on $k$-regular networks, where it arises as a result of the spontaneous symmetry breaking. All those facts are well known [13,15]. On the other hand, the permutation symmetry can be explicitly broken if the weight functions $p_i(m)$ are not identical for all nodes. For instance, this happens when the network on which the process takes place is inhomogeneous, that is, when the degrees $k_1, \ldots, k_N$ vary. A particularly important example are complex networks [16], for which the distribution of degrees has usually a long tail, and thus there are many nodes with relatively high degree. They are, however, not easy for analytical studies although some predictions are possible [17]. Below we shall argue that to gain some insight into the static and dynamical properties of the ZRP on such networks it is sufficient to study some simplified models.

In the remaining part of the paper we consider only the most favorable situation when the weight of only one node is different from the remaining ones. Such an inhomogeneity of the weights can be introduced either by an inhomogeneity of the outflow rates $u_i(m)$ [18] or by an inhomogeneity of the degree distribution. We focus here on the latter situation, when a graph has one node of degree $k_1$ which differs from all remaining degrees $k_2 = \cdots = k_N \equiv k$. As we shall see, in this case the quantity $\mu = \ln(k_1/k)$ plays the role of an external field breaking the permutation symmetry. A similar effect was studied in a one-dimensional ZRP with a single defect site [18] where the symmetry between the sites was explicitly broken by one site for which the hopping rate was smaller than for the others.

In particular, we discuss the dynamics of the condensate on such inhomogeneous networks. This is a relatively new topic and, in contrast to the stationary properties, less understood. Although the emergence of the condensate has been studied for homogeneous and inhomogeneous systems [13] and numerically for scale-free networks in [17], studies of the dynamics of an existing condensate are rare [19]. For instance, one question which may be asked is what is the typical lifetime of the condensate, that is, how much time does it take to "melt" the condensate at one node and rebuild it at another node. To provide an answer to this problem is the main goal of the present paper.

The rest of the paper is organized as follows. In Sec. II we discuss the static properties of the ZRP on inhomogeneous networks. We consider some particular graph topologies: $k$-regular graphs, star graphs, and $k$-regular graphs with a single inhomogeneity introduced by a vertex of degree $k_1 > k$. In all cases we calculate the effective occupation number distribution $\pi(m)$ and use it to derive information about the condensate dynamics. The advantage of studying ZRP on a single inhomogeneity network is that in this case one can do all calculations exactly or with a good, controlled approximation, similarly as in the model of Ref. [18]. On the other hand, despite its simplicity, the ZRP on a single inhomogeneity network captures the essence of the condensation dynamics observed on general inhomogeneous networks. In Sec. III we derive analytic expressions for the lifetime of the condensate. We concentrate on the role of inhomogeneity and typical scales at which it becomes relevant. All analytical results are cross-checked by Monte Carlo simulations. The last section is devoted to a summary of our results.

## II. STEADY STATE—STATICS

A zero-range process on a connected simple graph has a steady state with the following partition function $Z(N, M, \{k_i\})$ [13]:

$$Z(N,M,\{k_i\}) = \sum_{m_1=0}^{M} \cdots \sum_{m_N=0}^{M} \delta_{\sum_{i=1}^{N} m_i, M} \prod_{i=1}^{N} p(m_i) k_i^{m_i}, \qquad (1)$$

where $\delta_{i,j}$ denotes the discrete $\delta$ function and the weight function $p(m)$ is related to the hop rate $u(m)$ through the formula

$$p(m) = \prod_{n=1}^{m} \frac{1}{u(n)}, \quad p(0) = 1. \qquad (2)$$

We shall denote $Z(N, M, \{k_i\})$ in short by $Z(N, M)$, having in mind its dependence on the degrees. The partition function (1) contains the entire information about the static properties of the system in the steady state. The only trace of graph topology in the formula is through the nodes degrees. The dynamics, however, depends also on other topological characteristics, but they become important only in refined treatments. In a sense, the degree sequence is the first-order approximation also for the dynamics.

The probability $P(m_1, \ldots, m_N)$ of a given configuration $\{m_i\}$ reads as follows:

$$P(m_1, \ldots, m_N) = \frac{1}{Z(N,M)} \prod_{i=1}^{N} p(m_i) k_i^{m_i} = \frac{1}{Z(N,M)} \prod_{i=1}^{N} \tilde{p}_i(m_i), \qquad (3)$$

where we have defined "renormalized" weights $\tilde{p}_i(m_i) = p(m_i) k_i^{m_i}$, being now node-dependent. The most important quantity characterizing the steady state is the probability $\pi_i(m)$ that the $i$th node is occupied by $m$ particles,

$$\pi_i(m_i) = \sum_{m_1} \cdots \sum_{m_{i-1}} \sum_{m_{i+1}} \cdots \sum_{m_N} P(m_1, \ldots, m_N)\, \delta_{\sum_{j=1}^{N} m_j, M}$$
$$= \frac{Z_i(N-1, M-m_i)}{Z(N,M)} \tilde{p}_i(m_i), \tag{4}$$

where $Z_i(N-1, M-m)$ denotes the partition function for $M-m$ particles occupying a graph consisting of $N-1$ nodes with degrees $\{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_N\}$. We shall call $\tilde{p}_i(m)$ "bare" occupation probability while $\pi_i(m)$ "dressed" or effective occupation probability of the node $i$. We also define the average occupation probability

$$\pi(m) = (1/N) \sum_i \pi_i(m). \tag{5}$$

For a $k$-regular graph, that is, for $k_i \equiv k$, the occupation probability $\pi_i(m) = \pi(m)$ is the same for every node and all of the formulas above reduce to those discussed in [7]. In general, the partition function can be calculated recursively,

$$Z(N, M, \{k_1, \ldots, k_N\})$$
$$= \sum_{m_N} \tilde{p}_N(m_N) \sum_{m_1, \ldots, m_{N-1}} \delta_{\sum_{i=1}^{N-1} m_i, M-m_N} \prod_{i=1}^{N-1} \tilde{p}_i(m_i)$$
$$= \sum_{m_N=0}^{M} \tilde{p}_N(m_N) Z(N-1, M-m_N, \{k_1, \ldots, k_{N-1}\}). \tag{6}$$

For $N=1$ the partition function reads $Z(1, M, k_1) = \tilde{p}_1(M)$. The recursive use of the formula (6) allows one to compute the partition function within a given numerical accuracy. Using this method we were able to push the computation as far as to $N$ of order 500. For identical weights, one can find a more efficient recursion relation by splitting the system into two subsystems having similar size, which allows to study much larger systems. The computation of the partition function can be used together with Eq. (4) to determine numerically the node occupation distribution $\pi_i(m)$. This gives an exact result with a given accuracy and is more efficient than the corresponding Monte Carlo simulations of the ZRP. The dynamics, however, is not accessible in this way.

As mentioned in the introduction we examine the effect of topological inhomogeneity on the properties of the ZRP. We shall consider an almost $k$-regular graph with one node, say number one, having a degree bigger than the rest of the nodes, $k_1 > k = k_2 = \cdots = k_N$. The simplest realization of such a graph is a star or a wheel graph. In general, such a graph can be constructed from any $k$-regular graph by a local modification. We shall proceed as follows. First, we derive exact formulas for the particle distribution in a steady state on a $k$-regular graph which shall later serve us as a reference point. Then we consider the particular example of a star topology as the simplest example of a single defect, and finally an arbitrary $k$-regular graph with a singular node $k_1 > k$. The system has now the following weights: $\tilde{p}_1(m) = k_1^m p(m)$ for the singular and $\tilde{p}_i(m) = k^m p(m)$ for the regular nodes. They differ by an exponential factor $\tilde{p}_1(m)/\tilde{p}_i(m) = (k_1/k)^m = e^{\mu m}$, where $\mu = \ln(k_1/k) > 0$, which clearly favors the situation in which the singular node has many more particles than the regular ones similarly as in the one-dimensional model [18]. To make things as simple as possible, and to concentrate on the effect of inhomogeneity we assume that the outflow rate $u(m) = 1$ is constant and independent of $m$. In this case $p(m) = 1$ is also constant, simplifying calculations. All other functions with the asymptotic behavior $u(m) \to 1$ would lead basically to the same qualitative behavior. This is because in this case $p(m)$ would have a power-law tail which is much less important for the large $m$ behavior than the exponential factor $e^{\mu m}$ introduced by the inhomogeneity. We shall briefly comment on this toward the end of the paper.

### A. $k$-regular graph

With the assumption $p(m) = 1$, the partition function $Z(N, M)$ from Eq. (1) for the steady state of the ZRP on a $k$-regular graph reads as follows:

$$Z_{\mathrm{reg}}(N, M) = \sum_{m_1=0}^{M} \cdots \sum_{m_N=0}^{M} \delta_{\sum_i m_i, M} k^{\sum_i m_i}$$
$$= k^M \frac{1}{2\pi i} \oint \mathrm{d}z\, z^{-M-1} \left( \sum_{m=0}^{M} z^m \right)^N. \tag{7}$$

We used an integral representation of the discrete $\delta$ function which allowed us to decouple the sums over $m_1, \ldots, m_N$ for the price of having the integration over $z$. The sum over $m$ can be done yielding $1/(1-z)$. Using the expansion

$$\left( \frac{1}{1-z} \right)^N = \sum_{m=0}^{\infty} \binom{-N}{m} (-z)^m = \sum_{m=0}^{\infty} \binom{N+m-1}{m} z^m \tag{8}$$

and Cauchy's theorem we see that the contour integration over $z$ selects only the term with $m = M$ from the integrand in (7), so we obtain

$$Z_{\mathrm{reg}}(N, M) = k^M \binom{N+M-1}{M}. \tag{9}$$

Inserting this into Eq. (4) we find the occupation number distribution

$$\pi(m) = \pi_i(m) = \binom{M+N-m-2}{M-m} \Big/ \binom{M+N-1}{M}$$
$$\propto \frac{(M+N-m-2)!}{(M-m)!}. \tag{10}$$

The distribution $\pi(m)$ is identical for all nodes and independent on $k$. It falls faster than exponentially for large $m$, therefore, the condensate never appears.

In particular one can apply these formulas to the complete graph which is just a $k$-regular graph with $k = N-1$. In Fig. 1 we see the comparison between the theoretical expression (10) and results of numerical Monte Carlo simulations for a 4-regular graph with $N=20$ nodes and two different numbers of balls, $M$. The simulations of the ZRP were organized in sweeps consisting on $N$ steps each. In a single step a node was chosen at random and if it was non-empty a particle was picked and moved to a neighboring node. For each graph the

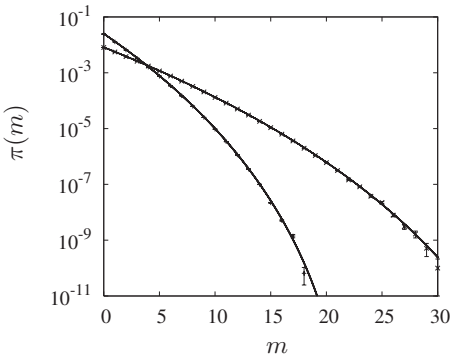


FIG. 1. The "experimental" distribution $\pi(m)$ compared to the theoretical prediction (10) for regular graphs with $k=4$ and $N=20$ and for $M=20$ (+) and $M=40$ (×) balls.

process was initiated from a random distribution of particles. After some thermalization, measurements of $\pi(m)$ were done on $10^4$ configurations generated every sweep. The results were averaged over these configurations and then over $5\times10^4$ independent graphs drawn at random from the ensemble of $k$-regular graphs.

### B. Star graph

We consider first a special case of a single inhomogeneity graph, namely the star graph having $N-1$ nodes of degree $k_2=\cdots=k_N=1$ connected to the central node with $k_1=N-1$. The partition function $Z_{\mathrm{star}}(N,M)$ is

$$Z_{\mathrm{star}}(N,M)=\sum_{m_1=0}^{\infty}\cdots\sum_{m_N=0}^{\infty}\delta_{\sum_{i=1}^{N}m_i,M}\,(N-1)^{m_1}$$
$$=\sum_{m=0}^{M}(N-1)^m\binom{M+N-m-2}{M-m}, \qquad (11)$$

as follows from Eq. (9). It is convenient to change the summation index from $m$ to $j=M-m$ which can be interpreted as a deficit of particles counted relatively to the full occupation,

$$Z_{\mathrm{star}}(N,M)=(N-1)^M\sum_{j=0}^{M}(N-1)^{-j}\binom{N+j-2}{j}. \qquad (12)$$

Let us assume that $N\gg1$. The summands in the last expression are strongly suppressed when $j$ increases so the sum can be approximated by changing the upper limit from $M$ to $\infty$. We obtain

$$Z_{\mathrm{star}}(N,M)\cong(N-1)^M\sum_{j=0}^{\infty}\left(\frac{-1}{1-N}\right)^j\binom{-(N-1)}{j}$$
$$=(N-1)^M\left(1-\frac{1}{N-1}\right)^{1-N}$$
$$=(N-1)^M\left(\frac{N-1}{N-2}\right)^{N-1}. \qquad (13)$$

Using Eq. (4) and the partition function $Z_{\mathrm{reg}}$ calculated in Eq. (9) of the preceding section we can determine the distribution of particles $m$ at the central (singular) node,

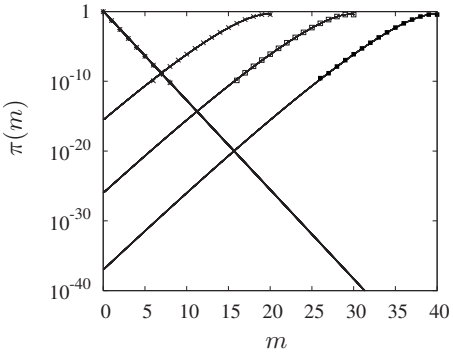


FIG. 2. The "experimental" and the theoretical (solid lines) particle number distributions for the star graph, for $N=20$ and $M=20$ (crosses), 30 (empty squares), and 40 (filled squares). The theoretical distributions were calculated according to the formula (14) for the central node (rising curves) and according to Eq. (15) for external nodes (falling curve, the Monte Carlo data (points) plotted for $N=20$ and $M=30$).

$$\pi_1(m)=\frac{Z_{\mathrm{reg}}(N-1,M-m)}{Z_{\mathrm{star}}(N,M)}(N-1)^m$$
$$=(N-1)^{m-M}\binom{M+N-m-2}{M-m}\left(\frac{N-2}{N-1}\right)^{N-1}. \qquad (14)$$

Similarly, we can determine the distribution of particles on any external (regular) node $i$,

$$\pi_i(m)=\frac{(N-2)(N-1)^{-m}}{N-1-(N-1)^{-M}}\approx\frac{N-2}{(N-1)^{m+1}}. \qquad (15)$$

We see that $\pi_i(m)$ decays exponentially with $m$ while $\pi_1(m)$ grows exponentially for $m\ll M$,

$$\pi_1(m)\propto e^{m\{-1/(2M)+1/[2(M+N-2)]+\ln[M(N-1)/(M+N-2)]\}}. \qquad (16)$$

The growth slows down for $m$ approaching $M$. At $m=M$, $\pi_1(m)$ reaches its maximal value

$$\pi_1(M)=\left(\frac{N-2}{N-1}\right)^{N-1}, \qquad (17)$$

which tends to $e^{-1}$ when $N$ goes to infinity. In Fig. 2 we compare the theoretical distributions (14) and (15) with Monte Carlo simulations of the ZRP for $N=20$ nodes and $M=20,30,40$. The agreement is very good.

It is instructive to calculate the mean number of particles at the central node. To simplify calculations we make use of the distribution $\hat{\pi}_1(j)\equiv\pi_1(M-j)$ of the deficit of particles defined above,

$$\hat{\pi}_1(j)=(1-\alpha)^{1-N}(-\alpha)^j\binom{-(N-1)}{j}. \qquad (18)$$

The parameter $\alpha=1/(N-1)$ is just the ratio of any external node degree to the degree of the central node and measures the level of inhomogeneity. The overall prefactor $(1-\alpha)^{1-N}$ is independent of $j$. It is just a normalization constant that

results from summing the $j$-dependent part of the expression (18),

$$S(\alpha) = \sum_{j=0}^{\infty} (-\alpha)^j \binom{-(N-1)}{j} = \frac{1}{(1-\alpha)^{N-1}}. \tag{19}$$

As before we changed the upper limit from $M$ to infinity because $\alpha \ll 1$ for the star graph and hence the summands are strongly suppressed for large $j$. The average deficit at the central node is

$$\langle j_1 \rangle = \sum_{j=0}^{M} \hat{\pi}_1(j) j = \alpha \frac{\mathrm{d} \ln S(\alpha)}{\mathrm{d}\alpha} = \frac{N-1}{N-2}, \tag{20}$$

as follows from Eq. (19). For large $N$ it tends to one, so we have

$$\langle m_1 \rangle = M - \langle j_1 \rangle \cong M - 1. \tag{21}$$

We see that on average almost all balls are concentrated at the central node and only one ball is in the rest of the system. We can also determine the range of fluctuations around $\langle m_1 \rangle$ by calculating the variance. Taking again advantage of the generating function (19) we find

$$\langle (m_1 - \langle m_1 \rangle)^2 \rangle = \langle (j_1 - \langle j_1 \rangle)^2 \rangle = \frac{\mathrm{d}^2 \ln S(e^{-\mu})}{\mathrm{d}\mu^2} = \left( \frac{N-1}{N-2} \right)^2, \tag{22}$$

where we used the parameter $\mu = -\ln \alpha = \ln(N-1)$. For large $N$ the result tends to one, so we can draw the following picture. For any $N \gg 1$ we observe a condensation of particles at the central node regardless of their density $\rho = M/N$. The critical density is equal to zero and the system is always in the condensed phase. The condensate residing at the central node contains $M-1$ particles, with very small fluctuations, while the other nodes are almost empty.

### C. Single inhomogeneity

A very particular property of the star graph is that the inhomogeneity increases with its size. Therefore, it is interesting to consider the situation when the inhomogeneity $\alpha = k/k_1$ is arbitrary and independent of $N$. The single inhomogeneity graph we consider here has one node of degree $k_1$ and $N-1$ nodes of degree $k$. Again, we assume that $k_1 > k$. The partition function (1) takes now the form

$$Z_{\mathrm{inh}}(N,M) = \sum_{m_1=0}^{M} k_1^{m_1} \sum_{m_2,\ldots,m_N=0}^{M} \delta_{\Sigma_i m_i, M} \, k^{\sum_{i=2}^{N} m_i}. \tag{23}$$

The sum over $m_2, \ldots, m_N$ is equal to the partition function $Z_{\mathrm{reg}}(N-1, M-m_1)$ given by Eq. (9). The whole formula looks almost identical to that for the star graph except that now the degree $k_1$ does not need to be much greater than $k$ and therefore the substitution of $M$ by $\infty$ must be done carefully in a manner incorporating finite-size corrections. As before, we first change variables from $m_1$ to $j = M - m_1$. Using Eq. (19) we can cast the formula (23) into the following form:

$$Z_{\mathrm{inh}}(N,M) = k_1^M \sum_{j=0}^{M} \alpha^j \binom{N+j-2}{j} = k_1^M [(1-\alpha)^{1-N} - c(M)]. \tag{24}$$

The correction $c(M)$ is equal to the sum over $j$ from $M+1$ to infinity. It corresponds to the surplus which must be subtracted from the infinite sum represented by the first term in square brackets. In the limit $M \to \infty$ it can be estimated as follows:

$$c(M) = \sum_{j=M+1}^{\infty} \alpha^j \binom{N+j-2}{j} \approx \frac{1}{(N-2)!} \int_M^{\infty} \mathrm{d}j e^{F(j)}, \tag{25}$$

where

$$F(j) = j \ln \alpha + \ln[(N+j-2)!] - \ln(j!). \tag{26}$$

Using Stirling's formula we can calculate the integral (25) by the saddle-point method. Taking into account only leading terms we have

$$\int_M^{\infty} \mathrm{d}j e^{F(j)} \approx e^{F(j_*)} \sqrt{\frac{-\pi}{2F''(j_*)}} \mathrm{erfc}\left[ (M - j_*) \sqrt{-F''(j_*)} \right], \tag{27}$$

where erfc denotes the complementary error function,

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-y^2} \mathrm{d}y, \tag{28}$$

and $j_*$ is determined from the equation $F'(j_*) = 0$,

$$j_* \approx \frac{\alpha(N-2)}{1-\alpha}. \tag{29}$$

Collecting all terms we eventually find

$$c(M) \approx \frac{\alpha^{\alpha(N-2)/(1-\alpha)}}{1-\alpha} \frac{[(N-2)/(1-\alpha)]!}{[\alpha(N-2)/(1-\alpha)]!} \sqrt{\frac{\pi\alpha(N-2)}{2}} \times \frac{1}{(N-2)!} \mathrm{erfc}\left( \frac{M(1-\alpha) - \alpha(N-2)}{\sqrt{\alpha(N-2)}} \right). \tag{30}$$

In order to keep formulas shorter we used here the notation $x! \equiv \Gamma(1+x)$ also for noninteger arguments. The complete partition function $Z_{\mathrm{inh}}(N,M)$ is given by the right-hand side of Eq. (24) with $c(M)$ given by Eq. (30). We can now calculate $\pi_1(m)$, that is the distribution of balls at the singular node,

$$\pi_1(m) = \frac{Z_{\mathrm{reg}}(N-1, M-m)}{Z_{\mathrm{inh}}(N,M)} k_1^m, \tag{31}$$

where $Z_{\mathrm{reg}}$ is the partition function (9) for a regular graph with degree $k$. Using Eqs. (9) and (24) we obtain

$$\pi_1(m) = \binom{M+N-m-2}{M-m} \frac{\alpha^{M-m}}{(1-\alpha)^{1-N} - c(M)}. \tag{32}$$

In Fig. 3 we show the theoretical distribution of balls for graphs with $k=4$, $N=20$, and various $M$ for singular nodes

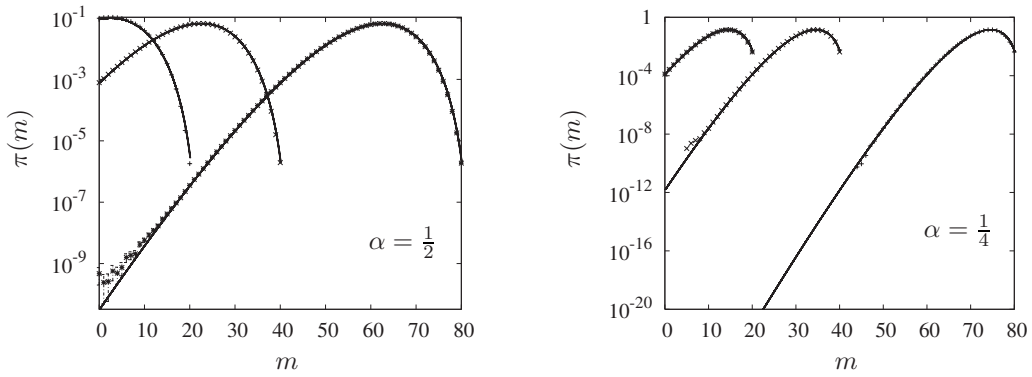


FIG. 3. The distribution of balls at the singular node for graphs with $k=4$, $N=20$, $k_1=8$ (left), and $k_1=16$ (right). The total number of balls is $M=20$, 40, and 80 from left to right curve, respectively. Points represent numerical data while solid lines show Eq. (32).

with $k_1=8$ and $k_1=16$, respectively, and compare them with the corresponding results obtained by Monte Carlo simulations. The agreement, which is already very good for the presented plots, is better, the smaller the ratio $\alpha=k/k_1$.

Neglecting an inessential normalization, we see that Eq. (32) has the asymptotic behavior

$$\pi_1(m) \propto \left(\frac{k_1}{k}\right)^m \binom{M+N-m-2}{M-m} \sim \exp[G(m)], \quad (33)$$

where

$$G(m) = \left(M+N-m-\frac{3}{2}\right)\ln(M+N-m-2) - m\ln\alpha - \left(M-m+\frac{1}{2}\right)\ln(M-m). \quad (34)$$

The number of particles of the condensate can be estimated using the saddle point equation $G'(m_*)=0$ for $m_*>0$. Neglecting terms of order $1/M^2$, we find

$$m_* \cong M - \frac{\alpha}{1-\alpha}(N-2). \quad (35)$$

Alternatively one can calculate the number of particles of the condensate as the mean value of the distribution $\pi_1(m)$. Adapting the same trick as in Eqs. (19) and (20) we obtain,

$$\langle m_1\rangle = M - \langle j_1\rangle = M - \alpha\frac{\mathrm{d}\ln S(\alpha)}{\mathrm{d}\alpha} = M - \frac{\alpha}{1-\alpha}(N-1) \approx m_*. \quad (36)$$

The criterion for condensation is that the central node contains an extensive number of balls. In the limit $N,M\to\infty$ and fixed density $\rho=M/N$ it amounts to the condition $\langle m_1\rangle>0$ leading to the critical density

$$\rho_c = \frac{\alpha}{1-\alpha}. \quad (37)$$

The condensation takes place when $\rho>\rho_c$ exactly like in the one-dimensional model [13,18]. The critical density decreases with decreasing ratio $\alpha=k/k_1$ or, equivalently, with increasing "external field" $\mu=\ln(k_1/k)$. The singular node attracts $N(\rho-\rho_c)+\rho_c$ balls on average as follows from Eq. (36). It is also easy to find that the distribution of balls $\pi_i(m)$ at any regular node falls exponentially,

$$\pi_i(m) \sim \left(\frac{k}{k_1}\right)^m = \alpha^m = e^{-\mu m}, \quad (38)$$

thus the condensate never appears on it. A regular node contains on average $\langle m_i\rangle=\rho_c$ balls independently of the total density of balls in the system $\rho$ as long as it exceeds $\rho_c$.

## III. DYNAMICS OF THE CONDENSATE

Let us now turn to a discussion of the dynamics of the condensate. From the preceding section we know that the condensate spends almost all time at the node with highest degree. However, occasionally it "melts" and disappears from the singular node for a short while. We know that the probability of such an event is very small, so we expect the lifetime of the condensate to be very large. Following the ideas of [19], let us imagine that we monitor only the number of particles at the singular node, which fluctuates in time. The temporal sequence of occupation numbers at this node performs a sort of one-dimensional random walk and can be viewed as a Markov chain. Using a mean-field approximation one can derive effective detailed balance equations for the incoming and the outgoing flow of particles for this node. The approximation is based on the assumption that the remaining part of the system gets quickly thermalized, much faster than the typical time scale of the melting process on the monitored node. Thus, the balance equations are written for the singular node and a single mean-field node having some typical properties. For the mean-field dynamics one can derive many quantities of interest. In particular, it is convenient to calculate the average time $\tau_{mn}$ it takes to decrease the occupation number of the monitored node from $m$ to $n$, or more precisely, the average first passage time for the Markov process initiated at $m$ to pass $n$. This quantity was first derived in [19] for the ZRP on a complete graph with outflow rates $u(m)=1+b/m$. The formula derived there,

$$\tau_{mn} = \sum_{p=n+1}^{m} \frac{1}{u(p)\pi(p)} \sum_{l=p}^{M} \pi(l), \quad (39)$$

can be easily adapted to the case discussed in our paper by setting $u(p)=1$ and using the distribution $\pi_1(p)$ of the singu-

lar node in place of $\pi(p)$ in the original formula. Equipped with the formula for $\tau_{mn}$ we are in principle able to calculate the typical melting time $\tau$. What is yet missing is the condition for $n$ at which the condensate can be considered as completely melted. We shall choose the simplest possible criterion and define the "typical" melting time $\tau$ as $\tau_{m0}$, that is the time needed to completely empty the monitored node beginning from $m$ equal to the average occupation of the node in the steady state.

It was shown in Ref. [19] that for the complete graph and $u(m)=1+b/m$, the melting time is approximately given by

$$\tau \propto (\rho-\rho_c)^{b+1}M^b, \tag{40}$$

where $\rho_c$ is the critical density above which the condensate is formed. The power-law increase with $M$ can be attributed to the power-law fall of $\pi(m)\sim m^{-b}$, characteristic for homogeneous systems with $u(m)=1+b/m$. The key point of our paper is that for inhomogeneous networks the melting time does no longer follow a power law but instead increases *exponentially* with $M$ due to the occurrence of the inhomogeneity which can be regarded as an external field $\mu=\ln(k_1/k)$, breaking the symmetry.

Before we perform the calculations let us make a general remark about the dependence of $\tau_{mn}$ on $m$ and $n$. A quick inspection of Eq. (39) tells us that a significant contribution to the sum over $p$ comes from terms for which $\pi(p)$, respectively, $\pi_1(p)$, is small. As we know from the preceding section, in the condensed phase $\pi_1(p)$ is many orders of magnitude greater for large $p$ than for small $p$. Therefore, when $m$ is of order $M$, and $n$ is of order 1, the time $\tau_{mn}$ varies very slowly with $m$ and, on the other hand, it is very sensitive to $n$. We thus set $m=M$ for simplicity and concentrate on $\tau_{Mn}$.

### A. Star graph

We assume $u(p)=1$ as before. Inserting the expression (14) for the particle occupation distribution $\pi_1(m)$ for the central node of the star into Eq. (39) in place of $\pi(m)$ we obtain

$$\tau_{mn}=\sum_{p=n+1}^{m}\sum_{l=p}^{M}(N-1)^{l-p}\frac{(M+N-l-2)\,!\,(M-p)!}{(M+N-p-2)\,!\,(M-l)!}. \tag{41}$$

From Sec. II B we know that the condensate contains $m=\langle m_1\rangle\approx M$ balls in the steady state and that fluctuations are very small. This justifies the choice $m=M$ we made above. Changing the summation variables similarly as in the preceding section we find

$$\tau_{Mn}=(N-2)\,!\sum_{r=0}^{M-n-1}\frac{r!}{(N-2+r)!}(N-1)^r \times\sum_{q=0}^{r}(N-1)^{-q}\frac{(N-2+q)!}{q\,!\,(N-2)!}. \tag{42}$$

In the second sum we can move the upper limit to infinity using exactly the same approximation as in Sec. II B,

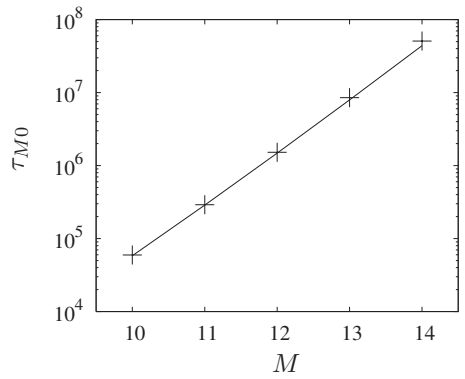


FIG. 4. The average lifetime $\tau_{M0}$ for the star graph with $N=10$ calculated from Eq. (45) (solid line) and found in computer simulations (points).

$$\tau_{Mn}\approx\left(\frac{N-1}{N-2}\right)^{N-1}(N-2)\,!\sum_{r=0}^{M-n-1}\frac{r\,!\,(N-1)^r}{(N-2+r)!}. \tag{43}$$

After a variable change $r\to M-n-1-r$, the remaining sum can be approximated as

$$\sum_{r=0}^{M-n-1}\frac{(M-n-1-r)!}{(M+N-n-3-r)!}(N-1)^{-r} \approx\frac{(M-n-1)!}{(M+N-n-3)!}\sum_{r=0}^{\infty}\left(\frac{M+N-n-3}{(M-n-1)(N-1)}\right)^r, \tag{44}$$

such that we finally arrive at the formula

$$\tau_{Mn}\approx\left(\frac{N-1}{N-2}\right)^{N}(N-2)\,!\,(N-1)^{M-n-1}\frac{M-n-1}{M-n-2} \times\frac{(M-n-1)!}{(M+N-n-3)!}. \tag{45}$$

We see that the presence of $(N-1)^{-n}$ makes the time $\tau_{Mn}$ indeed very sensitive to $n$. In Fig. 4 we see $\tau_{M0}$ compared to computer simulations. This complicated formula has a simple behavior in the limit of very large systems and for $n=0$. In the limit of large $M$ and for $N$ being fixed, the time $\tau_{Mn}$ grows exponentially with $M$,

$$\tau_{M0}\sim(N-1)^M=e^{\mu M}, \tag{46}$$

with $\mu=\ln(k_1/k)=\ln(N-1)$, while for fixed density $\rho=M/N$ and $N\to\infty$ it increases faster than exponentially,

$$\tau_{M0}\sim e^{\rho N\ln N}. \tag{47}$$

The approximate formulas (46) and (47) can be alternatively obtained using a kind of Arrhenius law [19,20], which states that the average lifetime is inversely proportional to the minimal value of the occupation number distribution,

$$\tau_{m0}\sim 1/\pi_1(\min), \tag{48}$$

where one thinks about the condensate's melting as of tunneling through the potential barrier in a potential $V(m)=-\ln\pi_1(m)$. In our case the potential $V(m)$ grows monotonically with $m$ going to zero, so the ball rather bounces from

the wall at $m=0$ than tunnels through it, but the reasoning is the same. From Eq. (14) we have $\pi_1(\min) \sim (N-1)^{-M}$ for fixed $N$ and large $M$ and we thus get again Eq. (46), while for fixed density $\pi_1(\min)$ falls over-exponentially which results in Eq. (47).

So far we have discussed the singular node. It is quite surprising that the formula (39) works also well for regular nodes. If we blindly substitute $\pi_1(k)$ by $\pi_i(k)$ we get the following expression for $\tau_{i,mn}$:

$$\tau_{i,mn} = \frac{(N-1)^n - (N-1)^m + (m-n)(N-2)(N-1)^M}{(N-2)^2(N-1)^{M-1}}. \quad (49)$$

For $m$ fixed, the transition time decreases almost linearly with $n$. The typical occupation of the regular node is much smaller than $M$, so we concentrate on $m \ll M, n=0$ and $N \gg 1$. The approximate formula reads as follows:

$$\tau_{i,m0} \approx m\frac{N-1}{N-2}. \quad (50)$$

We see that $\tau_{i,m0}$ grows very slowly in comparison to the lifetime of the condensate at the singular node. This linear growth can be easily understood as the minimal time needed for $m$ particles to hop out from a regular node. One must remember that in practice we cannot observe transitions for large $m$, because the probability of having such states is extremely small as it stems from Eq. (15) for $\pi_i(m)$.

### B. Single inhomogeneity

Next we consider the single inhomogeneity graph from Sec. II C. In the condensed phase the occupation $m$ fluctuates quickly around $\langle m_1 \rangle$ and even if it is smaller than $M$ we can assume that $\tau_{mn} \approx \tau_{Mn}$ because the transition time $\tau_{Mm}$ from $M$ to $m$ balls is very small in comparison to $\tau_{Mn}$ . Therefore, we shall concentrate again on $\tau_{Mn}$. From Eqs. (32) and (39) we have

$$\tau_{Mn} = \sum_{p=n+1}^{M} \sum_{l=p}^{M} \alpha^{p-l} \frac{\binom{M+N-l-2}{M-l}}{\binom{M+N-p-2}{M-p}}. \quad (51)$$

Changing variables we get

$$\tau_{Mn} = \sum_{p=n+1}^{M} \frac{(M-p)!}{(M+N-p-2)!} \sum_{q=0}^{M-p} \alpha^{-q} \frac{(M+N-p-q-2)!}{(M-p-q)!}. \quad (52)$$

The sum over $q$ can be approximated by an integral which can then be estimated by the saddle-point method. The saddle point is $q_* = \alpha(N-2)/(1-\alpha)$ as in Sec. II C and therefore all calculations are almost identical. In this way we obtain

$$\sum_q \cdots \approx \alpha^p \alpha^{\alpha[(N-2)/(1-\alpha)]-M} \frac{\left(\frac{N-2}{1-\alpha}\right)!}{\left(\frac{\alpha(N-2)}{1-\alpha}\right)!} \sqrt{\frac{2\pi\alpha(N-2)}{(1-\alpha)^2}}, \quad (53)$$

where we used again the notation $x! \equiv \Gamma(1+x)$. The only dependence on $p$ in this expression is through the factor $\alpha^p$.

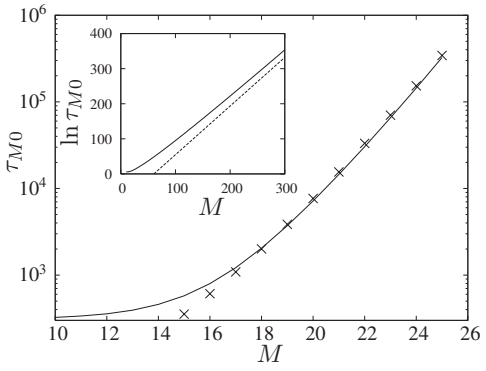


FIG. 5. Comparison between the "experimental" results (points) and the theoretical prediction (55) (solid line) for $\tau_{M0}$ of a 4-regular graph with single inhomogeneity $k_1=16$. The graph size is $N=20$. The inset compares $\tau_{M0}$ calculated from Eq. (55) (solid line) with that from Eq. (56) (dashed line), valid in the large-$M$ limit. The prefactor in Eq. (56) is chosen to match the $M \to \infty$ limit of the two formulas.

Thus to calculate $\tau_{Mn}$ it suffices to evaluate the sum

$$\sum_{p=n+1}^{M} \alpha^p \frac{(M-p)!}{(M+N-p-2)!}. \quad (54)$$

Because every term in the sum is proportional to $1/\pi_1(p)$ from Eq. (32), in the condensed phase the function under the sum has a minimum at the saddle point $p_*=m_*$. As this minimum is very deep, the effective contribution to the sum can be split into two terms: for small $p \ll m_*$ and for $p \gg m_*$. The "small-$p$" part can be evaluated as in Eq. (44) by pushing the upper limit to infinity and approximating the ratio of factorials by some number to the power $p$. To calculate the "large-$p$" part it is sufficient to take the last two terms in Eq. (54), namely for $p=M$ and $p=M-1$, because they decrease quickly. The complete formula for $\tau_{Mn}$ is finally given by

$$\tau_{Mn} \approx \alpha^{\alpha[(N-2)/(1-\alpha)]-M} \frac{\left(\frac{N-2}{1-\alpha}\right)!}{\left(\frac{\alpha(N-2)}{1-\alpha}\right)!} \sqrt{\frac{2\pi\alpha(N-2)}{(1-\alpha)^2}} \times \left[ \frac{M!}{(M+N-2)!} \left(\alpha \frac{M+N-2}{M}\right)^{n+1} \times \left(1-\alpha\frac{M+N-2}{M}\right)^{-1} + \frac{\alpha^{M-1}(\alpha(N-1)+1)}{(N-1)!} \right]. \quad (55)$$

In Fig. 5 we compare this theoretical formula with $\tau_{M0}$ from numerical simulations. Equation (55) simplifies in the limit of large systems. When one allows $M \to \infty$ while keeping $N$ and $\alpha$ fixed, then the lifetime grows exponentially,

$$\tau_{M0} \sim \left(\frac{1}{\alpha}\right)^M = e^{\mu M}. \quad (56)$$

For $\mu = \ln(N-1)$, that is for a star graph, it reduces to the formula (46). In the limit of fixed density $\rho = M/N > \rho_c$ and for $N, M \to \infty$,

$$\tau_{M0} \sim e^{N[-\ln(1-\alpha)+\rho\ln(\rho/\alpha)-(1+\rho)\ln(1+\rho)]}. \quad (57)$$

We see that the lifetime grows exponentially only if $k_1 > k$, that is, for positive external field $\mu = \ln(k_1/k)$. As before, we can explore the limit when the single inhomogeneity graph reduces to a star graph. Inserting $\mu = -\ln\alpha = \ln(N-1)$ into Eq. (57) we recover Eq. (47) as the leading term for large $N$.

## IV. CONCLUSIONS

In this paper we have studied static and dynamical properties of the condensation in zero-range processes on inhomogeneous networks. We have focused on the case where the network is almost a $k$-regular graph except that it has a single node of degree $k_1$ larger than $k$. This type of network could be a crude prototype of inhomogeneities encountered in scale-free networks having a single hub with very high degree and many nodes of much smaller degrees. Indeed, from the point of view of the hub, the remaining nodes look as if they formed an almost homogeneous system. We have shown that the distribution of balls $\pi_1(m)$ at the singular node has a maximum at $m \approx N(\rho - \rho_c)$, where $\rho_c$ is the critical density above which the condensate is formed. The average occupation of regular nodes is equal to $\rho_c$ and the condensate never appears on them. However, the condensate is not a static phenomenon. It fluctuates and it sometimes melts and disappears from the singular node. Then the surplus of balls distributes uniformly on all other nodes. After a while the condensate reappears and its typical lifetime $\tau$ grows exponentially like $e^{\mu M}$, where $M$ is the number of balls and $\mu = \ln(k_1/k)$ plays the role of an external field explicitly breaking the permutational symmetry of the system. This behavior is qualitatively distinct from that observed in homogeneous systems with a power-law distribution of balls, where $\tau$ grows only like a power of $M$, and the symmetry is spontaneously broken. Thus, the transition $\mu = 0 \to \mu > 0$ changes dramatically all of the properties of the system.

In all of the above calculations we assumed for simplicity that the hopping rate was $u(m) = 1$, and thus for the homogeneous system there would be no condensation. However, in the case of $u(m) = 1 + b/m$, which produces the power law in $\pi(m)$ for regular graphs, in an inhomogeneous system, apart from the condensation on the singular node, we would expect a second condensate on some regular node if the number of particles would be large enough to exceed the critical density for the homogeneous subsystem. Thus, we could expect the presence of two critical densities $\rho_{1c}$ and $\rho_{2c}$ and two condensates having completely distinct properties. We leave this interesting question for future research.

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